Forward-backward SDEs and the CIR model

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Abstract

We consider a forward-backward stochastic differential equation associated with the bond price for the Cox-Ingersoll-Ross interest rate model and prove an existence and uniqueness result. This technique is generalizable to multidimensional affine term structure models.

Key words: CIR model; bond price; forward-backward stochastic differential equations, Riccati equations

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1 Introduction

Cox, Ingersoll, and Ross (1985) proposed a model for the short rate which, Email address: hyndman@mathstat.concordia.ca (Cody Blaine Hyndman).

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provided certain parametric restrictions are imposed, remains nonnegative and includes stochastic volatility. On the risk neutral probability space \((\Omega, \mathcal{F}, Q)\) assume that the short rate, \(X_t\), has dynamics

\[
dX_t = \beta(\alpha - X_t)dt + \sigma \sqrt{X_t} dB_t
\]

with \(\alpha, \beta, \sigma > 0\) and \(2\beta \alpha > \sigma^2\). The Cox-Ingersoll-Ross (CIR) model has been extensively studied and has lead to generalizations in several directions. Duffie and Kan (1996), Filipović (2001), and Boyle et al. (2002) study the the CIR model and its generalizations and contain further references to the literature.

The price of the zero-coupon bond is given by

\[
P(t, T) = E_Q[\exp (- \int_t^T X_u du) | \mathcal{F}_t]
\]

at time \(t\) for maturity \(T\). For the CIR model the expectation in equation (1) can be solved in a number of different ways to obtain a well known closed-form solution for the bond price. The purpose of this paper is to provide an alternate approach, based on forward-backward stochastic differential equations (FBS-DEs, for short), to deriving the bond price. This approach is generalizable beyond the one dimensional case of the CIR model to the multidimensional case of the general square-root affine factors process studied by Duffie and Kan (1996) and Elliott and van der Hoek (2001).
2 Stochastic Flows and the Forward Measure

To begin we consider, as in Elliott and van der Hoek (2001), a stochastic
associated with the short rate process
\[ X_s^{t,x} = x + \int_s^t \beta(\alpha - X_v^{t,x})dv + \int_s^t \sigma \sqrt{X_v^{t,x}} dB_v \]
for all \(0 \leq t \leq s \leq T\) and \(x \in \mathbb{R}_{++}\). For all \(t \in [0, T]\), since \(X_t\) is a Markov
process, we have
\[ P(t, T) = P(t, T, X_t) \] (2)
\(Q - a.s\), where we define
\[ P(t, T, x) \equiv E_Q[\exp(-\int_t^T X_v^{t,x} dv)]. \] (3)
Write \((\partial_x X_s^{t,x})\) for the derivative of \(X_s^{t,x}\) with respect to the initial condition
\(x\). This derivative is well-defined and satisfies the linearized integral equation
\[ (\partial_x X_s^{t,x}) = 1 - \beta \int_t^s (\partial_x X_v^{t,x}) dv + \frac{\sigma}{2} \int_t^s \frac{(\partial_x X_v^{t,x})}{\sqrt{X_v^{t,x}}} dB_v. \] (4)
Differentiating equation (3) we obtain
\[ \partial_x P(t, T, x) = E_Q[\left(-\int_t^T \partial_x X_v^{t,x} dv \right) \exp(-\int_t^T X_v^{t,x} dv)] \]
subject to regularity conditions which allow the exchange of differentiation
and integration. By considering the forward measure \(P^T\), obtained by taking
the zero coupon bond price as numéraire, Elliott and van der Hoek (2001)
show that
\[ (\partial_x P(t, T, x))|_{x=X_t} = P(t, T, X_t) E_T[Y_t^{(B)} | \mathcal{F}_t] \] (5)
where \(E_T[]\) denotes expectation under \(P^T\) and \(Y_t^{(B)} = \left(-\int_t^T (\partial_x X_v^{t,x}) dv \right)|_{x=X_t}\).
Therefore, if the expectation under the forward measure of \((\partial_x X_v^{t,x})\) does not
depend on \(x\) the ordinary differential equation (5) can be solved to obtain an
exponential affine form for the bond price. The derivation given in Elliott and van der Hoek (2001) proceeds by showing that the process

$$B_t^T \triangleq B_t - \int_0^t E_T[Y_u^{(B)}|\mathcal{F}_u]\sigma \sqrt{X_u}du$$  \hspace{1cm} (6)

is a $\mathcal{F}_t$-Brownian motion with respect to the forward measure. Using equation (6) to write the dynamics (4) with respect to the forward measure and the fact that, for all $t \leq v$, $X_v^t, x_v = X_v$, it can be shown that the conditional expectation $\dot{D}_{ts} \triangleq E_T[(\partial_x X_v^t, x_v)|\mathcal{F}_t]$ satisfies

$$\dot{D}_{ts} = 1 - \beta \int_t^s \dot{D}_{tv}dv + \frac{\sigma^2}{2} \int_t^s E_T\left[\left(\partial_x X_v^t, x_v\right)\right]_{x=X_t} E_T[Y_v^{(B)}|\mathcal{F}_v]\mathcal{F}_t \right] dv$$  \hspace{1cm} (7)

for $0 \leq t \leq s \leq T$, almost surely.

Further, using the semigroup (or flow) property, the chain rule, and elementary properties of conditional expectation Elliott and van der Hoek (2001) show that $\dot{D}_{ts}$ satisfies the integral equation

$$\dot{D}_{ts} = 1 - \beta \int_t^s \dot{D}_{tv}dv + \frac{\sigma^2}{2} \int_t^s \int_v^T \dot{D}_{tv_2}dv_2dv_1, \quad 0 \leq t \leq s \leq T. \hspace{1cm} (8)$$

The integral equation (8) can be solved explicitly and, since the solution is independent of $\omega$, the term $E_T[Y_t^{(B)}|\mathcal{F}_t]$ is deterministic. The derivation given by Elliott and van der Hoek (2001) leads to the same bond price formula obtained by solving a Riccati equation (see Cox et al., 1985).

The techniques used by Elliott and van der Hoek (2001) to obtain equation (8) from equation (7) are not generalizable to the multidimensional square-root affine diffusion considered by Duffie and Kan (1996). As such, Elliott and van der Hoek (2001) consider a deterministic sequence of integral equations meant to approximate the multidimensional analog of the integral equation (7). However, there is a gap in the proof of the approximation result of
Elliott and van der Hoek (2001) inasmuch as an upper bound which is assumed to be constant actually grows at a rate which makes the application of Gronwall’s inequality ineffective. The original proof of Elliott and van der Hoek (2001) can be modified to provide a local result proving $\hat{D}_{ts}$ is deterministic for $t$ in a neighbourhood, $[T - \delta, T]$, of the terminal condition. Therefore, the characterization of bond prices as exponential affine functions of a factors process following a multidimensional square-root affine diffusion given by Elliott and van der Hoek (2001) is incomplete.

Our approach, which originated as an attempt to overcome the difficulty with the approximation result given by Elliott and van der Hoek (2001), provides a link with the results of Duffie and Kan (1996) and allows the same techniques used for the CIR model to be generalized to the case of a multidimensional square-root affine diffusion. We shall find that the adapted solution to a forward-backward stochastic differential equation can be related to the bond price and conditional expectation under $P^T$ of $Y_t^{(B)}$. These relationships allow us to prove, in a new way, that the conditional expectation under $P^T$ of $Y_t^{(B)}$ is deterministic. Further, our approach provides an alternative characterization of the bond price as exponential affine and has been further generalized to study the futures and forward price of a risky asset (see Hyndman, 2005).

3 The CIR model and bond price

Consider a FBSDE associated with the dynamics of the CIR interest rate model and the bond price:
\[ X_{s}^{t,x} = x + \int_{t}^{s} \beta(\alpha - X_{v}^{t,x})dv + \int_{t}^{s} \sigma \sqrt{X_{v}^{t,x}}dB_{v} \]

\[ Y_{s}^{t,x} = 1 - \int_{s}^{T} X_{v}^{t,x}Y_{v}^{t,x}dv - \int_{s}^{T} Z_{v}^{t,x}dB_{v} \]

for \( s \in [t, T] \). By variation of constants equation (10) has explicit solution

\[ Y_{s}^{t,x} = e^{-\int_{s}^{T} X_{v}^{t,x}dv} - \int_{s}^{T} e^{-\int_{s}^{v} X_{r}^{t,x}dr} Z_{v}^{t,x}dB_{v}. \] (11)

Similar to Proposition 4.2 of El Karoui et al. (1997) \( Y_{t}^{t,x} \) is deterministic, therefore, taking the expectation of equation (11)

\[ Y_{t}^{t,x} = E_{Q}[Y_{t}^{t,x}] = E_{Q}[e^{-\int_{t}^{T} X_{v}^{t,x}dv}]. \] (12)

Comparing equations (12) and (3), we have

\[ Y_{t}^{t,x} = P(t, T, x) \] (13)

and differentiating equation (13) with respect to \( x \)

\[ \frac{\partial}{\partial x} Y_{t}^{t,x} = \partial_{x} P(t, T, x). \] (14)

Therefore, since \( Y_{t}^{t,x} = P(t, T, x) > 0 \), equations (5), (13), and (14) give

\[ \left( \frac{\partial}{\partial x} Y_{t}^{t,x} \right) \bigg|_{x=X_{t}} = E_{T}[Y_{t}^{(B)}|\mathcal{F}_{t}]. \] (15)

Applying equation (15) to equation (6) gives

\[ B_{t}^{T} = B_{t} - \int_{0}^{t} \left( \frac{\partial}{\partial x} \frac{Y_{u}^{u,x}}{Y_{t}^{t,x}} \right) \bigg|_{x=X_{u}} \sigma \sqrt{X_{u}}du \] (16)

At this point we should like to apply Lemma 2.5 of Pardoux and Peng (1992) to express the integrand of (15) in terms of \( Y \) and \( Z \), Although the decoupled FBSDE (9)-(10) does not satisfy the hypotheses of Pardoux and Peng (1992) the result nonetheless holds. This can be justified by the Feynman-Kac Theorem and noting that if \( u(t, x) \) is a solution to the PDE
\[ \partial_t u(t, x) + \frac{1}{2} \sigma^2 x \partial_{xx} u(t, x) + \beta(x - x) \partial_x u(t, x) - xu(t, x) = 0 \]
\[ u(T, x) = 1 \]

then
\[ (Y_{s^i}^{t, x}, Z_{s^i}^{t, x}) \triangleq \left( u(t, X_{s^i}^{t, x}), \partial_x u(t, X_{s^i}^{t, x}) \sigma \sqrt{X_{s^i}^{t, x}} \right) \] (17)
satisfies the backward stochastic differential equation (BSDE, for short) (10).

A probabilistic argument, not relying on the solvability of a PDE, similar to that given by Pardoux and Peng (1992) for Lemma 2.5 of that paper would be preferable. Evaluating equation (17) at \((t, s, x) = (u, u, X_u)\) and substituting into equation (16) we find that the Brownian motion under the forward measure is given by
\[ B_t^T = B_t - \int_0^t \frac{Z_{u}^{u, X_u}}{Y_{u}^{u, X_u}} du. \] (18)

If we evaluate equations (9)-(10) at \(x = X_t\) then under the forward measure \((X_{s^i}^{t, x}, Y_{s^i}^{t, x}, Z_{s^i}^{t, x})\) satisfies
\[
X_{s^i}^{t, x_t} = X_t + \int_t^s \left\{ \beta(x - X_{v}^{t, x_t}) + \frac{Z_{v}^{v, X_v}}{Y_{v}^{v, X_v}} \sigma \sqrt{X_{v}^{t, x_t}} \right\} dv + \int_t^s \sigma \sqrt{X_{v}^{t, x_t}} dB_v^T
\]
\[
Y_{s^i}^{t, x_t} = 1 - \int_s^T \left\{ X_{v}^{t, x_t} Y_{v}^{t, x_t} + \frac{Z_{v}^{v, X_v}}{Y_{v}^{v, X_v}} \right\} dv - \int_s^T Z_{v}^{t, x_t} dB_v^T
\]
for \(s \in [t, T]\). By the flow property, \((X_{v}^{t, x_t}, Y_{v}^{t, x_t}, Z_{v}^{t, x_t}) = (X_{v}^{v, x}, Y_{v}^{v, x}, Z_{v}^{v, x})\) for all \(t \leq v \leq T\), we obtain the coupled nonlinear FBSDE
\[ X_{s^i}^{t, x_t} = X_t + \int_t^s \left\{ \beta(x - X_{v}^{t, x_t}) + \frac{Z_{v}^{t, X_v}}{Y_{v}^{t, X_v}} \sigma \sqrt{X_{v}^{t, x_t}} \right\} dv + \int_t^s \sigma \sqrt{X_{v}^{t, x_t}} dB_v^T \] (19)
\[ Y_{s^i}^{t, x_t} = 1 - \int_s^T \left\{ X_{v}^{t, x_t} Y_{v}^{t, x_t} + \frac{(Z_{v}^{t, X_v})^2}{Y_{v}^{t, X_v}} \right\} dv - \int_s^T Z_{v}^{t, x_t} dB_v^T. \] (20)

We shall drop the superscript \((t, X_t)\) and write \((X_s, Y_s, Z_s)\) for a solution of
the FBSDE (19)-(20). By Itô’s formula applied to equation (20) from $s$ to $T$

$$\log Y_s = - \int_s^T \left( X_v + \frac{1}{2} \frac{Z_v^2}{Y_v^2} \right) dv - \int_s^T \frac{Z_v}{Y_v} dB_v^T. \quad (21)$$

We next adapt a technique of Yong (1999) for linear FBSDEs to prove existence and uniqueness of the coupled nonlinear FBSDE (19)-(20).

**Theorem 3.1** If the Riccati equation

$$\dot{P}_B(t) - \beta P_B(t) + \frac{1}{2} \sigma^2 [P_B(t)]^2 - 1 = 0, \quad t \in [0, T] \quad (22)$$

$$P_B(T) = 0,$$

admits a solution $P_B(\cdot)$ then the FBSDE (19)-(20) admits a unique adapted solution $(X,Y,Z)$ given by

$$dX_s = \left( \beta(\alpha - X_s) + \sigma^2 P_B(s) X_s \right) ds + \sigma \sqrt{X_s} dB_s^T \quad (23)$$

$$\log Y_s = P_B(s) X_s + p_B(s) \quad (24)$$

$$Z_s = P_B(s) \sigma \sqrt{X_s Y_s} \quad (25)$$

where, for all $s \in [0, T]$

$$p_B(s) = \int_s^T \beta \alpha P_B(u) du. \quad (26)$$

**Proof:** Applying Itô’s formula from $s$ to $T$ to $f(s, x) = \exp \left( P_B(s) x + p_B(s) \right)$ when $X_s$ is given by the SDE (23) and $p_B(s)$ satisfies equation (26) gives that $Y_s = f(s, X_s)$ satisfies

$$Y_s = 1 - \int_s^T \left\{ \dot{P}_B(u) - \beta P_B(u) + \frac{1}{2} \sigma^2 [P_B(u)]^2 - 1 \right\} X_u Y_u du$$

$$- \int_s^T \left\{ \sigma^2 X_u [P_B(u)]^2 + X_u \right\} Y_u du - \int_s^T P_B(u) Y_u \sigma \sqrt{X_u} dB_u^T.$$

Applying equations (22) and (25) gives that $Y_s$ satisfies equation (20). Substituting equation (25) into equation (23) gives that $X_s$ satisfies equation (19).
Therefore, the process \((X, Y, Z)\) determined by equations (22), (23), (24), (25), and (26) is an adapted solution of the FBSDE (19)-(20).

To prove uniqueness let \((X, Y, Z)\) be any adapted solution of the FBSDE (19)-(20). Set

\[
\log Y_s = P_B(s)X_s + P_B(s)
\]

\[
\tilde{Z}_s = P_B(s)\sigma \sqrt{X_s} e^{P_B(s)X_s + p_B(s)}
\]

Applying Itô’s formula from \(s\) to \(T\) to the function \(f(s, x) = P_B(s)x + p_B(s)\) when \(X_s\) is given by the SDE (19) gives that \(f(s, X_s) = \log Y_s\) satisfies

\[
\log Y_s = - \int_s^T \left\{ \dot{P}_B(u) - \beta P_B(u) + \frac{1}{2} \sigma^2 [P_B(u)]^2 - 1 \right\} X_u du
\]

\[
- \int_s^T \left\{ X_u - \frac{1}{2} \sigma^2 [P_B(u)]^2 X_u + P_B(u)\sigma \sqrt{X_u} \frac{Z_u}{Y_u} \right\} du - \int_s^T P_B(s)\sigma \sqrt{X_u} dB_u^T
\]

\[
= - \int_s^T \left\{ X_u - \frac{1}{2} \frac{Z_u^2}{Y_u^2} + \frac{Z_u}{Y_u} \frac{Z_u}{Y_u} \right\} du - \int_s^T \frac{Z_u}{Y_u} dB_u^T.
\]

Therefore, by equation (21), we have

\[
\log Y_s - \log \tilde{Y}_s = - \int_s^T \frac{1}{2} \left( \frac{Z_u}{Y_u} - \frac{Z_u}{Y_u} \right)^2 du - \int_s^T \left( \frac{Z_u}{Y_u} - \frac{Z_u}{Y_u} \right) dB_u^T.
\]

Denote \(\hat{Y}_s = \log Y_s - \log \tilde{Y}_s\). We may set \(\tilde{Z}_s = (Z_u/Y_u - Z_u/Y_u)\) to get the following equivalent BSDE

\[
\hat{Y}_s = - \int_s^T \frac{1}{2} \frac{Z_u^2}{Y_u^2} ds - \int_s^T \tilde{Z}_u dB_u^T.
\]

By the results of Kobylanski (2000) such a BSDE admits a unique adapted solution \((\hat{Y}, \tilde{Z}) = (0, 0)\). This means that any adapted solution of the FBSDE (19)-(20) must satisfy (24)-(25). Then \(X\), given by (19)-(20), must also satisfy (23). Hence, we obtain uniqueness from the SDE (23).
Corollary 3.2 If the Riccati equation (22) admits a solution $P_B(\cdot)$ then

$$E_T[Y_t^{(B)}|\mathcal{F}_t] = P_B(t)$$

for $t \in [0, T]$.

Proof: If we evaluate equations (9)-(10) at $x = X_t$ then, under the forward measure, \{$(X^{t,X_t}_s, Y^{t,X_t}_s, Z^{t,X_t}_s); s \in [0, T]$\} satisfies the FBSDE (19)-(20). Since, by assumption, the Riccati equation (22) is solvable Theorem 3.1 implies that

$$\log Y^{t,X_t}_s = P_B(s)X^{t,X_t}_s + p_B(s), \quad s \in [t, T].$$

For $s = t$ we obtain $\log Y^{t,X_t}_t = P_B(t)X_t + p_B(t)$. Therefore,

$$E_T[Y_t^{(B)}|\mathcal{F}_t] = \left(\frac{\partial}{\partial x} Y^{t,x}_t\right)_{x=X_t} = \frac{\partial}{\partial X_t} \log Y^{t,X_t}_t = P_B(t).$$

\hfill \blacksquare

Corollary 3.3 If the Riccati equation (22) admits a solution $P_B(\cdot)$ over $[0, T]$ the bond price is exponential affine

$$P(t, T) = \exp \left(P_B(t)X_t + p_B(t)\right)$$

where $P_B(t)$ solves equation (22) and $p_B(t)$ solves equation (26).

Proof: By equations (2) and (13) $P(t, T) = P(t, T, X_t) = Y^{t,X_t}_t$ and, as in the proof of Corollary 3.2, \{$(X^{t,X_t}_s, Y^{t,X_t}_s, Z^{t,X_t}_s); s \in [0, T]$\} satisfies the FBSDE (19)-(20). Since, by assumption, the Riccati equation (22) is solvable Theorem 3.1 implies that $\log Y^{t,X_t}_t = P_B(t)X_t + p_B(t)$.

\hfill \blacksquare

Corollary 3.4 If the Riccati equation (22) admits a solution $P_B(\cdot)$ over $[0, T]$ then $\hat{D}_{tu}$ is deterministic for $0 \leq t \leq u \leq T$
Proof: If the Riccati equation (22) admits a solution $P_B(\cdot)$ over $[0, T]$ we have, by Corollary 3.2, that $P_B(v_1) = E_T[Y_{v_1}^{(B)}|\mathcal{F}_{v_1}]$ so from equation (7) we obtain

$$
\dot{D}_{tu} = 1 - \beta \int_t^u \dot{D}_{tv_1} dv_1 + \frac{\sigma^2}{2} \int_t^u P_B(v_1) E_T[(\partial_x X_{v_1}^t X_{v_1})|\mathcal{F}_t] dv_1
$$

$$
= 1 + \int_t^u \left\{ -\beta + \frac{\sigma^2}{2} P_B(v_1) \right\} \dot{D}_{tv_1} dv_1
$$

almost surely. This is equivalent to a deterministic (time-varying) linear ordinary differential equation. Let $\Psi(u)$ be the solution to the linear o.d.e.

$$
x'(u) = \left\{ -\beta + \frac{\sigma^2}{2} P_B(u) \right\} x(u).
$$

Then $\Phi(u, t) = \Psi(u) \Psi^{-1}(t)$ for $0 \leq t \leq u \leq T$ solves

$$
\Phi(u, t) = 1 + \int_t^u \left\{ -\beta + \frac{\sigma^2}{2} P_B(v) \right\} \Phi(v, t) dv.
$$

Hence $\dot{D}_{tu} = \Phi(u, t)$ is deterministic. $
$
It is well known that the Riccati equation (22) has solution

$$
P_B(t) = -\frac{2(e^{(T-t)} - 1)}{\beta - \gamma - (\beta + \gamma)e^{(T-t)}}
$$

t \in [0, T]$ so that

$$
p_B(t) = \int_t^T \beta \alpha P_B(s) ds = \frac{2\alpha\beta}{\gamma} \log \left[ \frac{2\gamma e^{(\beta+\gamma)(T-t)/2}}{(\beta + \gamma)e^{\gamma(T-t)} + \gamma - \beta} \right].
$$

For $t \in [0, T]$ we may define

$$
B(t, T) \triangleq -P_B(t) = \frac{2(e^{(T-t)} - 1)}{\beta - \gamma - (\beta + \gamma)e^{(T-t)}}
$$

$$
A(t, T) \triangleq p_B(t) = \int_t^T \beta \alpha P_B(s) ds = \frac{2\alpha\beta}{\gamma} \log \left[ \frac{2\gamma e^{(\beta+\gamma)(T-t)/2}}{(\beta + \gamma)e^{\gamma(T-t)} + \gamma - \beta} \right]
$$

where $\gamma = \sqrt{\beta^2 + 2\sigma^2}$. Then $P(t, T) = \exp(A(t, T) - B(t, T)X_t)$ which agrees
with the results of Cox et al. (1985) and the derivation presented in Elliott and van der Hoek (2001).

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References


