Supplement to "Two (Talking) Heads Are Not Better than One"

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Proof of Proposition 1. Since we consider symmetric equilibria, $y_{-1} = -y_1$ and $y_0 = 0$. It must be that $y_1 > 0$ otherwise the equilibrium is uninformative. Let $y = y_1$ to save notation. Since y_s and s have the same sign, we have $-(y_s - s)^2 < -(y_{s'} - s)^2$ for all $s, s' \in S, s \neq s'$. Therefore, $m_0^*(s) = s$ for all $s \in S$.

I calculate $m_1^*(s)$ only and $m_{-1}^*(s)$ follows from symmetry. Note that $b_1 = b \in [\frac{17}{21}, \frac{6}{7}]$

First, it is straightforward to see $m_1^*(1) = 1$.

Second, $m_1^*(0) = 1$. Since $|0-(0+b)| > |y-(0+b)|$, and $|-y-(0+b)| > |0-(0+b)|$ for all $b > \frac{1}{2}$ and $y \in (0, 1]$, we have $u(1, 0, b) = -(y-(0+b))^2 > -(0-(0+b))^2 > -(-y-(0+b))^2$.

Finally, $m_1^*(-1) \neq 1$ because $b < 1$ implies $|0-(-1+b)| < |y-(-1+b)|$. If $m_1^*(-1) = -1$ (by symmetry, $m_{-1}^*(1) = 1$), then $y = \frac{P(s=1)\cdot 1 + P(s=0,x=1)\cdot 0}{P(s=1)+P(s=0,x=1)} = (1/3)/(1/3 + 1/3 \times 1/3) = 3/4$. Thus $|0-(-1+b)| = 1-b < b-\frac{1}{4} = |-y-(-1+b)|$, which makes $m_1(-1) = -1$ not optimal. The inequality comes from the fact that $b \geq \frac{17}{21} > \frac{5}{8}$. So only $m_1^*(-1) = 0$ can be part of the equilibrium strategy. In this case, $y = \frac{P(s=1,x\neq -1)\cdot 1 + P(s=0,x=1)\cdot 0}{P(s=1,x\neq -1)+P(s=0,x=1)} = (1/3 \times 2/3)/(1/3 \times 2/3 + 1/3 \times 1/3) = 2/3$, thus $|0-(-1+b)| = 1-b < b-\frac{1}{3} = |-y-(-1+b)|$. Again the inequality comes from the fact that $b \geq \frac{17}{21} > \frac{2}{3}$. Hence $m_1^*(-1) = 0$ is optimal. To summarize, $m_1^*(s) = s+1$ if $s \neq 1$, and $m_1^*(1) = 1$, and by symmetry $m_{-1}^*(s) = s-1$ if $s \neq -1$, $m_{-1}^*(-1) = -1$.

In the above paragraph, I have shown that $y_1^* = 2/3$. Furthermore the decision maker's expected utility is

$$
\begin{aligned}
&-[P(s=1, x\neq -1)(y^*-1)^2 + P(s=-1, x\neq 1)(-y^*-(-1))^2 \\
&+P(s=1, x=-1)(0-1)^2 + P(s=-1, x=1)(0-(-1))^2
\end{aligned}
$$

$$
\begin{aligned}
& +P(s=0,x=1)(y_1^*-0)^2+P(s=0,x=-1)(-y_1^*-0)^2] \\
= \; & -2[2/9\times(1/3)^2+1/9\times 1^2+1/9\times(2/3)^2] \\
= \; & -10/27 \qquad \square
\end{aligned}
$$

Proof. (of Proposition 2) The proof is by checking incentive conditions. Since the strategy profile is symmetric, between a strategy and its mirror image, it is sufficient to check the optimality of one of them.

It is straightforward to see that the decision maker's actions are optimal conditional on the experts' strategies. Consider, for example, the probability that the message pair $(1,-1)$ is received and the state is s is respectively $\frac{3}{27}$, $\frac{2}{27}$, and $\frac{2}{27}$ for $s=-1$, 0, and 1. Thus, the conditional expectation of s given that $(1,-1)$ is received should be

$$y(1,-1)=\frac{3}{7}\cdot(-1)+\frac{2}{7}\cdot 0+\frac{2}{7}\cdot 1=-\frac{1}{7}.$$

Now, I proceed to check the optimality of Expert B's strategies. First note that by assumption,

$$b\in(\frac{47}{70},\frac{13}{14})\subset(\frac{2}{3},1).$$

Most of the incentive conditions are clear by inspection. I only verify in details an arguably less obvious condition, i.e., that $m_1^B(s,-1)=1$ is optimal for $s=-1$, 0, and 1. When $s=-1$, the expert's most preferred action is $-1+b$. But

$$|(-1+b)-y(-1,1)|=\left|(-1+b)-(\frac{1}{7})\right|<\left|(-1+b)-(-\frac{4}{5})\right|=|(-1+b)-y(-1,-1)|,$$

where the inequality is due to the fact that $b>\frac{47}{70}$. Thus a positive-biased Expert B finds it optimal to send the message 1. For $s=0$, it is clear that the message 1 is optimal, since the the expert's most preferred action is b, and since

$$y(-1,-1)<0<y(-1,1)<b.$$

Lastly, in state 1, the positive-biased expert's most preferred action is $1+b$, and a condition similar to the one immediately above holds. Therefore, $m_1^B(s,-1)=1$ is optimal for $s=-1$, 0, and 1.

Now, I check the optimality of Expert A's strategies. Again, I check the arguably least obvious condition, i.e., $m_1^A(-1)=1$. A positive-biased Expert A can send message -1, 0, or 1 in state -1. Given Expert B's strategies, the corresponding outcomes and probabilities are listed below:

$$\begin{array}{ll} m_A = -1 & \frac{2}{3} \cdot y(-1,-1) \bigoplus \frac{1}{3} \cdot y(-1,1) = \frac{2}{3} \cdot (-\frac{4}{5}) \bigoplus \frac{1}{3} \cdot \frac{1}{7} \\ m_A = 0 & 1 \cdot y(0, m_B) = 1 \cdot 0 \\ m_A = 1 & 1 \cdot y(1,-1) = 1 \cdot (-\frac{1}{7}) \end{array}$$

The expression $p \cdot y \bigoplus q \cdot y'$ indicates that the variable takes value y with probability p, and value y' with probability q. Since $b < 13/14$, it is clear that sending 1 dominates sending 0. On the other hand, both $-\frac{4}{5}$ and $\frac{1}{7}$ is worse than $-\frac{1}{7}$ in state -1 for the positive-biased Expert A, since in state -1 the order of preference for the positive-biased expert is $-\frac{1}{7}$, 0, $\frac{1}{7}$, and $-\frac{4}{5}$. Therefore, sending 1 also dominates sending -1.

Straightforward calculation shows that the decision maker's expected payoff is $-44/105$. □

Proof. Detailed argument for why a large enough payment for contradictions (message pairs $(1,-1)$ or $(-1,1)$) breaks down equilibria of Li (2004).

Let β be the additional payoff each expert gets if $(-1,1)$ or $(1,-1)$ is sent.

For equilibrium (B), as $y(-1, m_B) = -2/3$ for all m_B, to ensure it to be an equilibrium, only the response 1 can be sent when the first message is -1, as Expert B has a strict incentive to say 1 to "contradict" Expert A. Given this, as long as the following condition is satisfied, an Expert A of type 1 does not find it optimal to say 0 in state -1 as saying -1 will give her a higher payoff:

$$\beta > \frac{2}{3}\left(\frac{2}{3}b - \frac{8}{9}\right).$$

This condition is obtained from comparing her payoff from sending -1,

$$-((-1+b) - (-\frac{2}{3}))^2 + \beta,$$

and that from sending 0

$$-\frac{2}{3}((-1+b) - (-\frac{2}{3}))^2 - \frac{1}{3}((-1+b) - 0)^2.$$

For equilibria (C) and (D), a type-1 Expert B finds it optimal to say -1 instead of 1 in state 0 if

$$\beta > \frac{3}{5}b - \frac{39}{100}.$$

This condition is obtained from comparing her payoff from sending 1 (thereby inducing $y(1,1) = 4/5$) and that from sending -1 (thereby inducing $y(1,-1) = 1/2$). $\square$

References

Li, M. (2004): "Combining Expert Opinions," *mimeo, Concordia University, available at http://alcor.concordia.ca/˜mingli/research/combine.pdf.*