Asset Pricing Models

Econ 643: Financial Economics II

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Stochastic Discount Factor Approach

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- Unifying approach to pricing stocks, bonds and derivative products
  - (1) is referred to as fundamental pricing equation (Cochrane, 2001).

Stochastic Discount Factor Approach

- Dividing both sides of (1) by p_t (assuming non-zero prices) and rearranging

  $$E_t(m_{t+1}R_{t+1} - 1) = 0$$

  or equivalently

  $$E(\, m_{t+1}R_{t+1} - 1 |\, I_t) = 0$$

  where $R_{t+1} = \frac{x_{t+1}}{p_t} = \frac{p_{t+1}+d_{t+1}}{p_t}$ denotes the gross asset return.

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- In this case, with zero price and payoff $R^e_{t+1} = R_{t+1} - R^f_{t+1}$, the fundamental pricing equation is given by
$$E_t(m_{t+1}R^e_{t+1}) = 0.$$

Stochastic Discount Factor Approach: An Example

- Suppose that a representative agent maximizes expected utility

$$\sum_{t=1}^{\infty} \beta^t E[u(c_t)|I_0]$$

subject to a budget constraint

$$a_{t+1} = (a_t + y_t - c_t)\, R_{t+1},$$

where c_t is consumption, a_t is an asset with gross return R_t, y_t is income and I_t is the information set at time t.

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- It takes the form of the fundamental pricing equation with the SDF $m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$ given by the intertemporal marginal rate of substitution.

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- But imposing explicitly the no-arbitrage constraint may have some adverse effects.
- More on how and whether we should impose positivity of the SDF, see Gospodinov, Kan and Robotti (2010).

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- $\beta_{i,m} = Cov(m, R^i)/Var(m)$ is the regression coefficient of the return R^i on m and $\lambda_m = -Var(m)/E(m) < 0$ is the price of risk.

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- Some interesting observations emerge from rewriting

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  - these assets provide insurance against bad outcome and smooth consumption.

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  - γ and λ are the same for all assets.

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- If the factors are also excess returns (e.g., market excess returns), then $\lambda_j = E(f^j)$ and

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- The model implies $E(R^{ei}) = \beta_i E(f)$ and the pricing errors should be jointly equal to zero.

Evaluation of Factor Models

- To test if the model is correctly specified (i.e. the pricing errors are zero), use the statistic

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- If the pricing errors are large, the test statistic will exceed the critical value from the chi-square distribution with N degrees of freedom and the null hypothesis will be rejected.

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  2. Second pass (cross-sectional regression): estimating the factor risk premia λ and pricing errors α from the equation

  $$\overline{R}^{ei} = \widehat{\beta}_i' \lambda + \alpha_i, \;\; i = 1, 2, ..., N$$

  as $\widehat{\lambda} = (\widehat{\beta}' \widehat{\beta})^{-1} \widehat{\beta}' \overline{R}^e$ (OLS) or $\widehat{\lambda} = (\widehat{\beta}' \widehat{\Sigma}^{-1} \widehat{\beta})^{-1} \widehat{\beta}' \widehat{\Sigma}^{-1} \overline{R}^e$ (GLS) and $\widehat{\alpha} = \overline{R}^e - \widehat{\lambda}' \widehat{\beta}$, where $\overline{R}^{ei} = \frac{1}{T} \sum_{t=1}^{T} R_t^{ei}$.

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  - we will see later that this correction does not need to be performed explicitly if the model is estimated by GMM.

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- We will consider first the statistical approach based on factor analysis and principal components.

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  - FF three-factor model plus term structure factor (difference in the yields on 30-year bond and 1-month bill) and default premium (difference between the yields on BAA and AAA corporate bonds).

Conditional Asset Pricing Models

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- The conditional asset pricing model that satisfies $p_t = E_t(m_{t+1}x_{t+1})$ is the model

$$m_{t+1} = a_t + b_t' f_{t+1},$$

where a_t and b_t are possibly time-varying.

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  - i.e., instead of a one-factor model with time-varying coefficients, we have a three-factor model with fixed coefficients.

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- When we have many factors and instruments, we obtain the scaled factors by multiplying each factor by each instrument using the Kronecker product

$$\textit{scaled factors} = f_{t+1} \otimes z_t.$$

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    - compare its performance (using Sharpe ratio) to a buy-and-hold benchmark strategy over the out-of-sample evaluation period.