**Question 1**. A direct application of Ito's lemma yields

$$\frac{d(S/P)}{S/P} = (\mu_S - \mu_P - \rho\sigma_S\sigma_P + \sigma_P^2)dt + \sigma_S dz_S - \sigma_P dz_P.$$

If $\rho = 0$, the drift term is $\mu_S - \mu_P + \sigma_P^2$ and is greater than $\mu_S - \mu_P$ (expected nominal drift of the stock $\mu_S$ minus expected inflation $E(dP/P)$). The extra term $\sigma_P^2$ is a Jensen's inequality term that arises from the convexity of $S/P$ in $P$.

**Question 2**. If $Y(t) = X(t)^2$, $\mu = 0$ and $\sigma = 1$, then

$$\frac{\partial f}{\partial t} = 0, \frac{\partial f}{\partial X} = 2X(t), \frac{\partial^2 f}{\partial X^2} = 2$$

and by Ito's lemma,

$$\begin{aligned} dY(t) &= \frac{1}{2}2dt + 2X(t)dz(t) \\ &= dt + 2z(t)dz(t) \end{aligned}$$

since $dX(t) = dz(t)$ and hence, $X(t) = z(t)$.

Integrating from $t_0$ to $t$ gives

$$\int_{t_0}^{t} dY(s) = \int_{t_0}^{t} ds + 2\int_{t_0}^{t} z(s)dz(s)$$

or

$$z(t)^2 - z(t_0)^2 = t - t_0 + 2\int_{t_0}^{t} z(s)dz(s)$$

since $\int_{t_0}^{t} dY(s) = \int_{t_0}^{t} dz(s)^2 = z(t)^2 - z(t_0)^2$.

Therefore,

$$\int_{t_0}^{t} z(s)dz(s) = \frac{1}{2}\left[z(t)^2 - z(t_0)^2\right] - \frac{t - t_0}{2}.$$

**Question 3**. Applying Ito's lemma to $S = e^X$ yields

$$dS = \mu S dt + \sigma S dz,$$

where $\mu = (\alpha + \sigma^2/2)$. Since $S$ follows a GBM, we already know from class that $\ln(S_t/S_0)$ is normally distributed. Let $y = \ln(S_T/S_0)$. Then, under risk neutrality,

$$y \sim N\left(m, \sigma_y^2\right),$$

where $m = (r - \sigma^2/2)T$ and $\sigma_y^2 = \sigma^2 T$.

Using the definition of expectation and the formula for the p.d.f. of a log-normally distributed random variable (see Green's econometrics textbook, for example),

$$\begin{aligned} E(S_T \,|S_T > X\,) &= \int_X^\infty S_T f(S_T) dS = S_0 \int_X^\infty e^y f(S_T) dS \\ &= S_0 \int_X^\infty e^y \frac{1}{S\sigma_y\sqrt{2\pi}} e^{-\frac{[\ln S - (\ln S_0 + m)]^2}{2\sigma_y^2}} dS \\ &= \frac{S_0}{\sigma_y\sqrt{2\pi}} \int_X^\infty e^y e^{-\frac{(y-m)^2}{2\sigma_y^2}} \frac{dS}{S} \\ &= \frac{S_0}{\sigma_y\sqrt{2\pi}} \int_{\ln(X/S_0)}^\infty e^{y - \frac{(y-m)^2}{2\sigma_y^2}} dy \end{aligned}$$

since $dy = dS/S$ and $S > X$ is equivalent to $y > \ln(X/S_0)$.

Some manipulations of the integrand give

$$
\begin{aligned}
\exp\left[y - \frac{(y-m)^2}{2\sigma_y^2}\right] &= \exp\left[-\frac{1}{2\sigma_y^2}\left\{(y-m)^2 - 2y\sigma_y^2\right\}\right] \\
&= \exp\left[-\frac{1}{2\sigma_y^2}\left\{y^2 - 2ym + m^2 - 2y\sigma_y^2\right\}\right] \\
&= \exp\left[-\frac{1}{2\sigma_y^2}\left\{y^2 - 2y(m+\sigma_y^2) + (m+\sigma_y^2)^2 - (2m\sigma_y^2 - \sigma_y^4)\right\}\right] \\
&= \exp\left[-\frac{(y-m^*)^2}{2\sigma_y^2}\right]\exp\left[m + \frac{\sigma_y^2}{2}\right],
\end{aligned}
$$

where $m^* = m + \sigma_y^2$.

Substituting this back into the integral yields

$$E(S_T|S_T > X) = S_0 e^{m+\sigma_y^2/2}\int_{\ln(X/S_0)}^{\infty} \frac{1}{\sigma_y\sqrt{2\pi}} e^{-\frac{(y-m^*)^2}{2\sigma_y^2}}\, dy,$$

where the last integral term represents the probability that a normal variable $y$ with mean $m^*$ and variance $\sigma_y^2$ is larger than $\ln(X/S_0)$.

Then,

$$
\begin{aligned}
\Pr\left(y > \ln(X/S_0)\right) &= \Pr\left(\frac{y-m^*}{\sigma_y} > \frac{\ln(X/S_0) - m^*}{\sigma_y}\right) \\
&= \Pr\left(z > \frac{\ln(X/S_0) - m - \sigma_y^2}{\sigma_y}\right) \\
&= \Pr\left(z > \frac{\ln(X/S_0) - (r - \sigma^2/2)\tau - \sigma^2\tau}{\sigma\sqrt{\tau}}\right),
\end{aligned}
$$

where $z$ is a standard normal random variable.

Since $\Pr(z > -d_1) = 1 - N(-d_1) = N(d_1)$,

$$
\begin{aligned}
E\left(S_T|S_T > X\right) &= S_0 e^{m+\sigma_y^2/2} N(d_1) \\
&= S_0 e^{r\tau} N(d_1)
\end{aligned}
$$

using that $m + \sigma_y^2/2 = m + \sigma^2\tau/2 = r\tau$.

Thus,

$$e^{-r\tau} E\left(S_T|S_T > X\right) = S_0 N(d_1).$$

**Question 4.**

(a) The options that are most sensitive to changes in volatility should maximize $\frac{\partial C}{\partial \sigma}$ which is the "vega" of the option

$$\nu = \frac{\partial C}{\partial \sigma} = S\sqrt{\tau}N'(d_1),$$

where $N'(d_1)$ is the standard normal p.d.f.

Since $S$ and $\tau$ are positive constants, the vega is maximized where $N'(d_1)$ is maximized. Using that $N'(d_1)$ is a standard normal density, the maximum occurs when the argument is equal to the mean or $d_1 = 0$. Then,

$$\ln(S/X) + (r + \sigma^2/2)\tau = 0$$

or, by exponentiating both sides and rearranging,

$$\frac{e^{-r\tau}X}{S} = e^{-\sigma^2\tau/2}.$$

This is satisfied by close-to-maturity ($\tau \approx 0$), at-the-money (moneyness $\approx 0$) options.

(b) A sufficient condition for the Black-Scholes formula to be linear in volatility is $\frac{\partial^2 C}{\partial \sigma^2} = 0$. In part (a) we argued that this occurs when $d_1 = 0$. Then, near-maturity, at-the-money options make the BS formula linear in volatility. This implies that the implied volatility extracted from these option prices should be an unbiased forecast of future volatility.

(c) Taking the first derivative of the "vega" with respect to $\tau$ yields

$$\begin{aligned} \frac{\partial \nu}{\partial \tau} &= \nu\left(\frac{rd_1}{\sigma\sqrt{\tau}} - \frac{1+d_1d_2}{2\tau}\right) \qquad (1) \\ &= \nu\left(\frac{r}{2} - \frac{1-\tau\sigma^2/4}{2\tau}\right) \end{aligned}$$

using that for at-the-money options

$$d_1 = \frac{\ln(S_t/X) + (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} = \frac{\ln(S_t/e^{-r\tau}X) + \sigma^2\tau/2}{\sigma\sqrt{\tau}} = \frac{\sigma\sqrt{\tau}}{2}$$

and

$$d_2 = d_1 - \sigma\sqrt{\tau} = -\frac{\sigma\sqrt{\tau}}{2}.$$

The derivative in (1) is equal to zero when the term in brackets is zero or

$$\tau^* = \frac{1}{r+\sigma^2/4}.$$

**Question 7**. Computing the implied volatilities at different maturities produces the following graph.