Question 1. Suppose the price level follows the diffusion process

$$dP/P = \mu_P dt + \sigma_P dz_P$$

and the stock price follows

$$dS/S = \mu_S dt + \sigma_S dz_S,$$

with correlation between dz_P and dz_S of ρ. Use Ito's lemma to find the dynamics of the real stock price S/P. If $\rho = 0$, is the drift in the expected real stock price is more or less than $\mu_S - \mu_P$? Explain why.

Question 2. Suppose we want to evaluate the stochastic integral $\int_{t_0}^{t} z(s)dz(s)$, where $z(s)$ is the Wiener process. The usual rules of integration of ordinary calculus do not apply to this integral although they still can be used for the integrals $\int_{t_0}^{t} s dz(s)$ and $\int_{t_0}^{t} z(s)ds$.

(a) Let $Y(t) = X(t)^2$ and $dX(t) = dz(t)$ (i.e. $\mu = 0$ and $\sigma = 1$). Use Ito's lemma to find that $dY(t) = dt + 2z(t)dz(t)$.

(b) Integrate the equation from part (a). Use the usual rules of ordinary calculus to find $\int_{t_0}^{t} dY(s)$ using the definition of $Y(t)$.

(c) Use the results from part (b) to show that $\int_{t_0}^{t} z(s)dz(s) = \frac{1}{2}\left[z(t)^2 - z(t_0)^2\right] - \frac{1}{2}(t - t_0)$.

Question 3. Let X follow an arithmetic Brownian motion $dX = \alpha dt + \sigma dz$ and $X = \ln(S)$, where S is the price of a stock that pays no dividends. The risk-free interest rate is denoted by r. First show that the stock price follows $dS = \mu S dt + \sigma S dz$ and write down the expression for μ under risk neutrality. What is the distribution of $\ln(S_T/S_0)$ conditional on the current value S_0? What is the mean and the variance of this distribution? Using this result, prove that $e^{-rT}E\left(S|S > X\right) = S_0 N(d_1)$, where X is the exercise price, $N(.)$ is the standard normal distribution function and $d_1 = \frac{\ln(S_0/X)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$.

Question 4. Let $C = BS(S, X, \tau, r, \sigma)$ denote the price of a European call option on this stock, where $BS(.)$ is the Black-Scholes formula derived in class.

(a) Suppose that we want to estimate the implied volatility but at each time point we have a large cross-section of option prices with different exercise prices (and hence moneyness defined as $\left[e^{-r\tau}X\right]/S - 1$) and maturities. We would like to use options that contain the most information about volatility, i.e. options that are most sensitive to changes in volatility. Find, in terms of moneyness and maturities, the options that are most informative about volatility and discuss the result.

(b) Find the moneyness and the maturity that make the Black-Scholes formula linear in volatility. Discuss the result.

(c) Finally, find the maturity at which the "vega" of the call option is maximized for at-the-money options (i.e., $e^{-r\tau}X/S - 1$).

Question 5. In this question, you need to evaluate the accuracy of Ito's lemma. Suppose that X follows an arithmetic Brownian motion, $dX = .1dt + .25dz$ with initial value $X_0 = 2$ and $f(X) = X^2$. For $\Delta t = .1, .01, .001, .0001$, calculate the mean and the standard deviation of the error from Ito's approximation of Δf as well as the ratio of the standard deviation of the error to the standard deviation of Δf.

Hint: Construct a grid of 999 equispaced points from 0.001 to 0.999. Use these points as an argument in the inverse standard normal distribution function (`normsinv` in Excel, `cdfni` in GAUSS, `invnorm` in STATA) to obtain random numbers from $N(0,1)$. Multiplying these numbers by $\sqrt{\Delta t}$ gives possible values for dz. For each value of dz, compute $X(t+\Delta t)$, $f[X(t+\Delta t)]$, the actual Δf, the approximation of Δf from Ito's lemma and the error resulting from Ito's approximation, $\Delta f(actual) - \Delta f(Ito)$.

Question 6. Go to `http://finance.yahoo.com/q/hp?s=%5EGSPC` and download daily, weekly and monthly quotes (close) for S&P500 index from January 1960 to December 2010.

(a) At each frequency, calculate the log returns (multiplied by 100) and then compute their mean, standard deviation, skewness and kurtosis coefficients (see pages 16-17 in the book by Campbell, Lo and MacKinlay). Are the skewness and kurtosis statistically different from those of the standard normal distribution? Comment on the validity of our assumption that stock prices follow a geometric Brownian motion. As you go from daily to monthly data, does the distribution of the S&P500 returns approach the standard normal distribution or not? What is the rate at which the standard deviation increases as you go from daily to weekly and then to monthly returns?

(b) Use the daily data to compute and plot the realized and range-based monthly volatility for S&P500 log returns. Construct the corresponding standardized returns and compute their descriptive statistics. Discuss the results.

Question 7. Consider a stock that follows a geometric Brownian motion with current market price $S = 100$. The risk-free rate is 8%. Suppose that you observe the following market information about the prices of call options with different maturities and strike prices on this stock.

strike X	1 month call price	3 months call price	6 months call price
90	11.39	14.48	18.25
95	7.48	11.10	15.15
100	4.45	8.27	12.43
105	2.40	6.01	10.11
110	1.18	4.28	8.16
115	0.54	3.00	6.55
120	0.24	2.08	5.23

Calculate the implied volatilities for each entry in the table. Plot the implied volatilities as a function of moneyness defined as $\left[e^{-r(T-t)}X\right]/S - 1$. Plot the implied volatility curves for the three maturities on the same graph. Comment on the shape of the implied volatility across different moneyness and maturity.