# Solutions to Assignement 4

(10.2) Lagging the equation for $gGDP$ one period gives

$$gGDP_{t-1} = \alpha_0 + \delta_0 \, inf_{t-1} + \delta_1 \, inf_{t-2} + u_{t-1}$$

Plug this into the second equation

$$inf_t = \gamma_0 + \gamma_1(\alpha_0 + \delta_0 \, inf_{t-1} + \delta_1 \, inf_{t-2} + u_{t-1} - 3) + v_t$$

$$= (\gamma_0 + \gamma_1\alpha_0 - 3\gamma_1) + \gamma_1\delta_0 \, inf_{t-1} + \gamma_1\delta_1 \, inf_{t-2} + \gamma_1 u_{t-1} + v_t$$

$$Cov(inf_t, u_{t-1}) = E(inf_t \, u_t) = \gamma_1 E(u_{t-1}^2) > 0 \text{ if } \gamma_1 > 0$$

This violates the strict exogeneity assumption TS.2

(11.4)

$$y_t = e_t + e_{t-1} + \dots + e_1$$

$$y_{t+h} = e_{t+h} + e_{t+h-1} + \dots + e_1$$

$$E(y_t) = E(y_{t+h}) = 0$$

$$Var(y_t) = t\sigma_e^2 \qquad Var(y_{t+h}) = (t+h)\sigma_e^2$$

$$Cov(y_t, y_{t+h}) = E(y_t \, y_{t+h}) = E[(e_t + e_{t-1} + \dots + e_1)(e_{t+h} + e_{t+h-1} + \dots$$

$$= E(e_t^2) + E(e_{t-1}^2) + \dots + E(e_1^2) = t\sigma_e^2$$

since $e_t$ is uncorrelated.

$$Corr(y_t, y_{t+h}) = \frac{Cov(y_t, y_{t+h})}{\sqrt{Var(y_t)\,Var(y_{t+h})}} = \frac{t\sigma_e^2}{\sqrt{t\sigma_e^2 (t+h)\sigma_e^2}} = \sqrt{\frac{t}{t+h}}$$

## Additional Problem 1

(a) for unbiasedness of the OLS estimator we need the stronger condition $E(u_t | X) = 0$
The OLS estimator is consistent because
TS.1', TS.2' ($E(u_t | x_t) = 0$) and TS.3' are satisfied
Since $Var(u_t | x_t) = \sigma_u^2 x_t^2$ (heteroskedasticity)
and $E(u_t u_s | x_t, x_s) \neq 0$ (serial correlation)
OLS estimator is not BLUE

(b) We can not estimate directly $h_t = \exp(\delta_0 + \delta_1 x_t^2 + \delta_2 \frac{1}{x_t})$ by OLS since it is nonlinear in the parameters.
After taking logs on both sides
$$\ln(h_t) = \delta_0 + \delta_1 x_t^2 + \delta_2 \frac{1}{x_t} + u_t$$
which is linear in the parameters (although it is non-linear in the variables) and can be estimated by OLS
A consistent and more efficient estimator is FGLS
① estimate (1) by OLS and get the residuals $\hat{u}_t$
② Run a regression of $\ln(\hat{u}_t^2)$ on a constant, $x_t^2$, $\frac{1}{x_t}$ and get the predicted values $\widehat{\ln(\hat{u}_t^2)}$.
③ $\hat{h}_t = \alpha_0 \exp(\widehat{\ln(\hat{u}_t^2)})$ where $\alpha_0$ is a correction factor
④ Estimate $\beta_0$ and $\beta_1$ by OLS from
$$\frac{y_t}{\sqrt{\hat{h}_t}} = \beta_0 \frac{1}{\sqrt{\hat{h}_t}} + \beta_1 \frac{x_t}{\sqrt{\hat{h}_t}} + \frac{u_t}{\sqrt{\hat{h}_t}}$$

(c) substitute for $y_t$, $x_t = \alpha_0 + \alpha_1(\beta_0 + \beta_1 x_t + u_t) + e_t$
Since $E(x_t u_t) = \frac{\alpha_1}{(1-\alpha_1\beta_1)} \sigma_u^2 \neq 0$ (assuming that $e_t$ is independent of $u_t$)
the OLS estimator of $\beta_1$ in (1) is inconsistent.
If we find a variable $z_t$ such that $E(u_t | z_t) = 0$ and $corr(x_t, z_t) \neq 0$, we can use the instrumental variables estimato

## Additional Problem 2

(a) $E(y_t) = E(e_t + \theta e_{t-1}) = E(e_t) + \theta E(e_{t-1}) = 0$

since $E(e_t) = E(e_{t-1}) = 0$

$$\mathrm{Var}(y_t) = E(y_t^2) = E(e_t^2 + \theta^2 e_{t-1}^2 + 2\theta e_{t-1} e_t)$$
$$= \underset{\sigma^2}{\underset{\shortparallel}{E(e_t^2)}} + \theta^2 \underset{\sigma^2}{\underset{\shortparallel}{E(e_{t-1}^2)}} + 2\theta \underset{0}{\underset{\shortparallel}{E(e_{t-1} e_t)}} \quad \text{by assumption}$$
$$= (1+\theta^2)\sigma^2$$

$$\mathrm{Cov}(y_{t-1}, y_t) = E(y_{t-1} y_t) = E[(e_{t-1} + \theta e_{t-2})(e_t + \theta e_{t-1})]$$
$$= \underset{0}{\underset{\shortparallel}{E(e_{t-1} e_t)}} + \theta \underset{\sigma^2}{\underset{\shortparallel}{E(e_{t-1}^2)}} + \theta \underset{0}{\underset{\shortparallel}{E(e_{t-2} e_t)}} + \theta^2 \underset{0}{\underset{\shortparallel}{E(e_{t-2} e_{t-1})}}$$
$$= \theta \sigma^2$$

For any $k > 1$, $E(y_{t-k} y_t) = 0$

Therefore, $y_t$ is covariance stationary (none of the above expressions depend on $t$) and weakly dependent ($\mathrm{Corr}(y_{t-k}, y_t) \to 0$ as $k \to \infty$)

(b) Since $u_t = y_t - \beta y_{t-1}$,

$u_{t-1} = y_{t-1} - \beta y_{t-2}$ and

$u_{t-2} = y_{t-2} - \beta y_{t-3}$

Substitute for $u_{t-1}$ and $u_{t-2}$ into the second equation of model (2)

$$u_t = \rho_1(y_{t-1} - \beta y_{t-2}) + \rho_2(y_{t-2} - \beta y_{t-3}) + e_t$$

and then substitute for $u_t$ into the first equation of model (2)

$$y_t = \beta y_{t-1} + \rho_1(y_{t-1} - \beta y_{t-2}) + \rho_2(y_{t-2} - \beta y_{t-3}) + e_t$$

$$= \underbrace{(\beta + \rho_1)}_{\alpha_1} y_{t-1} + \underbrace{(\rho_2 - \rho_1 \beta)}_{\alpha_2} y_{t-2} \underbrace{- \rho_2 \beta}_{\alpha_3} y_{t-3} + e_t$$

The OLS estimator of $\beta$ in (2) is <u>inconsistent</u> because 2 relevant variables ($y_{t-2}$ and $y_{t-3}$) have been omitted.

(c) $$y_t = \delta_0 x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + u_t$$

$H_0: \delta_0 + \delta_1 + \delta_2 = 1$

Define $\theta = \delta_0 + \delta_1 + \delta_2$ and $\delta_0 = \theta - \delta_1 - \delta_2$

Substituting for $\delta_0$ into the equation gives

$$y_t = (\theta - \delta_1 - \delta_2) x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + u_t$$

$$= \theta x_t - \delta_1 (x_t - x_{t-1}) - \delta_2 (x_t - x_{t-2}) + u_t$$

Running a regression of $y_t$ on $x_t$, $x_t - x_{t-1}$ and $x_t - x_{t-2}$ produces $\hat{\theta}$ and s.e.$(\hat{\theta})$ and we can use these to compute $t_{\hat{\theta}}$ and test

$H_0: \theta = 1$