Solutions for Midterm Exam

(Q1)

(a) discussion on sign and magnitude

$\Delta sleep = -.148(5 \times 60) = -44.4$ minutes per week

not a large effect

(b) $t_{educ} = \dfrac{-11.13}{5.88} \approx -1.89$

$t_{age} = \dfrac{2.20}{1.45} \approx 1.52$

crit. value at 5% sign. level is 1.96

$|t_{educ}| < 1.96$ and $|t_{age}| < 1.96$

both variables are individually insignificant at 5% s.l.

(c) $F = \dfrac{(0.113 - 0.103)/2}{(1 - 0.113)/702} \approx 3.96 > 3$ (crit. value from F-table at 5% s.l.)

educ and age are jointly significant

(d) the explanatory variables in (2) explain only 11.3% of the variation in sleep.

Other factors: health, marital status, children etc.

These omitted variables are likely to be correlated with totwrk and age and their omission will result in inconsistency of the OLS estimator

(e) The marginal effect of age on sleep is

$$\frac{\partial\, sleep}{\partial\, age} = \beta_3 + 2\beta_4\, age$$

If $age = 30$, the null hypothesis that the marginal effect of age is 0 is $H_0: \beta_3 + 60\beta_4 = 0$ or $\theta = 0$ where $\theta = \beta_3 + 60\beta_4$. t-test $= \dfrac{\hat\beta_3 + 60\hat\beta_4}{s.e.(\hat\beta_3 + 60\hat\beta_4)}$ but we don't know $Cov(\hat\beta_3, \hat\beta_4)$. Instead, we rewrite the model with $\beta_3 = \theta - 60\beta_4$ and estimate $\hat\theta$ from the transformed model.

(Q2) (a) „partialling out" interpretation of $\hat{\beta}_1$

First, we partial out the effect of x_2 by running a regression of x_1 on x_2 and computing the residuals $\hat{r}_1$ that gives us the part of x_1 that is uncorrelated with x_2

Next, we estimate $\hat{\beta}_1$ from a regression of y on $\hat{r}_1$ which measures the ceteris paribus effect of x_1 on y after x_2 has been partialled out.

(b)
$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} \hat{r}_{i1} y_i}{\sum_{i=1}^{n} \hat{r}_{i1}^2} = \frac{\sum \hat{r}_{i1}(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + u_i)}{\sum \hat{r}_{i1}^2} =$$

$$= \beta_0 \frac{\sum \hat{r}_{i1}}{\sum \hat{r}_{i1}^2} + \beta_1 \frac{\sum \hat{r}_{i1}(\hat{x}_{i1} + \hat{r}_{i1})}{\sum \hat{r}_{i1}^2} + \beta_2 \frac{\sum \hat{r}_{i1} x_{i2}}{\sum \hat{r}_{i1}^2} + \frac{\sum \hat{r}_{i1} u_i}{\sum \hat{r}_{i1}^2}$$

$$= 0 + \beta_1 \frac{\sum \hat{r}_{i1} \hat{x}_{i1}}{\sum \hat{r}_{i1}^2} + \beta_1 + 0 + \frac{\sum \hat{r}_{i1} u_i}{\sum \hat{r}_{i1}^2}$$

$$= \beta_1 + \frac{\sum \hat{r}_{i1} u_i}{\sum \hat{r}_{i1}^2}$$ using the algebraic properties of OLS residuals that they sum up to 0, they are uncorrelated with the regressors x_2 and the fitted values $\hat{x}_1$.

Taking expectations of both sides, conditional on x_1 and x_2 gives

$E(\hat{\beta}_1) = \beta_1$ since $\hat{r}_1 = x_1 - \hat{x}_1$ is non-random conditional on x_1 and $E(u_i | x_{i1}, x_{i2}) = 0$

(c) Taking variances of both sides of the above expression, conditional on x_1 and x_2 gives

$$\mathrm{Var}(\hat{\beta}_1) = \left[\frac{1}{\sum \hat{r}_{i1}^2}\right]^2 \mathrm{Var}\left[\sum \hat{r}_{i1} u_i\right] = \frac{\sigma^2 \cancel{\sum \hat{r}_{i1}^2}}{\left[\sum \hat{r}_{i1}^2\right]^2} = \frac{\sigma^2}{SSR_1}$$

Using that $SSR_1 = (1 - R_1^2)\, SST_1$,

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{(1-R_1^2)\, \Sigma (x_{i1} - \bar{x}_1)^2}$$

(d) $\hat{\beta}_1$ and $\tilde{\beta}_1$ are equivalent if (1) $\hat{\beta}_2 = 0$
or (2) x_1 and x_2 are uncorrelated in the sample

Generally, $\tilde{\beta}_1$ is expected to have lower variance because

$$Var(\tilde{\beta}_1) = \frac{\sigma^2}{\Sigma (x_{i1} - \bar{x}_1)^2} \leq \frac{\sigma^2}{(1-R_1^2)\, \Sigma (x_{i1} - \bar{x}_1)^2} = Var(\hat{\beta}_1)$$

since $0 \leq R_1^2 < 1$.