FURTHER RESULTS ON THE LIMITING DISTRIBUTION OF GMM SAMPLE MOMENT CONDITIONS

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Supplementary Material

SIMULATION SETUP

This appendix contains some additional simulation results regarding the properties of the standard normal test, the weighted χ^2 test, the LM rank test, and the sequential test considered in the paper. In the simulation experiment, the factors ($\mathbf{f}$) and the returns ($\mathbf{R}$) on the test assets for the CAPM (1 factor and 11 test asset returns) and FF3 (3 factors and 26 test asset returns) are drawn from a multivariate normal distribution with a covariance matrix estimated from the data. The mean return vector is chosen such that the asset pricing model holds exactly for the test assets. For each simulated set of returns and factors, the unknown parameters $\boldsymbol{\theta}_0$ of the linear SDF $y(\boldsymbol{\theta}_0) = \tilde{\mathbf{f}}'\boldsymbol{\theta}_0$, where $\tilde{\mathbf{f}} = (1,\ \mathbf{f}')'$, are estimated by minimizing the sample HJ-distance, which yields

$$\hat{\boldsymbol{\theta}} = (\hat{\mathbf{D}}_T'\mathbf{W}_T\hat{\mathbf{D}}_T)^{-1}(\hat{\mathbf{D}}_T'\mathbf{W}_T\mathbf{q}), \tag{1}$$

where $\hat{\mathbf{D}}_T = \frac{1}{T}\sum_{t=1}^{T}\mathbf{R}_t\tilde{\mathbf{f}}_t'$, $\mathbf{W}_T = \left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{R}_t\mathbf{R}_t'\right)^{-1}$, and $\mathbf{q} = [1,\ \mathbf{0}_{m-1}']'$. The estimated Lagrange multipliers are given by

$$\hat{\boldsymbol{\lambda}} = \mathbf{W}_T\left[\frac{1}{T}\sum_{t=1}^{T}\mathbf{R}_t y_t(\hat{\boldsymbol{\theta}}) - \mathbf{q}\right], \tag{2}$$

where $y_t(\hat{\boldsymbol{\theta}}) = \tilde{\mathbf{f}}_t'\hat{\boldsymbol{\theta}}$.

We consider linear combinations of sample Lagrange multipliers with different choices of an $m \times 1$ nonzero weighting vector $\boldsymbol{\alpha}$, i.e., $\boldsymbol{\alpha}'\hat{\boldsymbol{\lambda}}$. Let matrix $\mathbf{Q}_c$ denote the null space of the p vector $E[\tilde{\mathbf{f}}_t\tilde{\mathbf{f}}_t']\boldsymbol{\theta}_0$ and $\mathbf{Q}_c^1$ be the first column of $\mathbf{Q}_c$. Also, let $\boldsymbol{\Pi} = \mathbf{P}_{\boldsymbol{\alpha}}'\mathbf{D}_0$, where $\mathbf{P}_{\boldsymbol{\alpha}}$ is an $m \times (m-1)$ orthonormal matrix whose columns are orthogonal to $\boldsymbol{\alpha}$. In Tables I through IV, we analyze the empirical sizes of four tests – (i) standard normal test of $H_0 : \boldsymbol{\alpha}'\boldsymbol{\lambda} = 0$, (ii) weighted χ^2 test of $H_0 : \boldsymbol{\alpha}'\boldsymbol{\lambda} = 0$, (iii) LM rank test of $H_0 : \text{rank}(\boldsymbol{\Pi}) = p-1$, and (iv) sequential test of $H_0 : \boldsymbol{\alpha}'\boldsymbol{\lambda} = 0$ with a pre-test of $H_0 : \text{rank}(\boldsymbol{\Pi}) = p-1$, using three choices of $\boldsymbol{\alpha}$:

1. $\boldsymbol{\alpha} = \mathbf{q} = [1\ , \mathbf{0}_{m-1}']'$,
2. $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{1}_p$,
3. $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{Q}_c^1$.

We also analyze the statistical properties of the rank and sequential tests when $\boldsymbol{\alpha}$ in not in the span of the column space of $\mathbf{D}_0$. Specifically, in Table V, we analyze the empirical power of the rank test for $\boldsymbol{\alpha} = \mathbf{1}_m$ and $\boldsymbol{\alpha} = \sqrt{m}\mathbf{q} + \mathbf{1}_m$. In Table VI, we report results for the empirical size of the sequential test for $\boldsymbol{\alpha} = \mathbf{1}_m$ and $\boldsymbol{\alpha} = \sqrt{m}\mathbf{q} + \mathbf{1}_m$. The empirical rejection probabilities are computed based on 100,000 Monte Carlo replications.

STANDARD NORMAL TEST

Panels A and B of Table I show that the use of the standard normal test leads to severe over-rejections when $\boldsymbol{\alpha}$ is in the span of the column space of $\mathbf{D}_0$. By contrast, the normal test behaves well in Panel C. These simulation results can be explained using the theoretical results in Lemma 6 in the paper. In particular, in Panel A we have $\boldsymbol{\alpha} = \mathbf{q}$ and $r = -1$, and the t-test is asymptotically distributed as $-\sqrt{\chi^2_{m-p}}$. In Panel B, the squared t-test follows a mixture of two independent chi-squared random variables with $m-p$ and one degrees of freedom. Finally, in Panel C, $\boldsymbol{\alpha}$ is set such that $r_2 = 0$ (and $r = 0$) and the t-test follows a standard normal distribution which explains why the t-test works well in this setup.[1]

WEIGHTED χ^2 TEST

In Table II, we report the empirical size of the weighted χ^2 test. For the CAPM, our asymptotic approximation works very well even for relatively small sample sizes. For FF3, we need a larger T for the asymptotic approximation to work well. This is a well-known problem in empirical asset pricing that arises when the number of test assets m is large relative to T (see, e.g., Ahn and Gadarowski, 2004).

RANK TEST

Tables III and V report the empirical size and power of the rank test. Overall, the test has excellent size and power properties. Some modest under-rejections only occur for FF3 when $T =$ 150.

SEQUENTIAL TEST

In Tables IV and VI, we analyze the empirical size of the sequential test (that includes a reduced rank pre-test) of $H_0 : \lambda_1 = 0$ when $\boldsymbol{\alpha}$ is in the span of the column space of $\mathbf{D}_0$ and when $\boldsymbol{\alpha}$ is not. The sequential test we consider has the following structure. If we reject the null of reduced rank,

[1] Note that our conclusions are not affected by the particular choice of the column of $\mathbf{Q}_c$ (the matrix described in the simulation setup).

then we use the normal test in the second stage; otherwise, we use the weighted chi-squared test. Acceptance and rejection of $H_0 : \boldsymbol{\alpha}'\boldsymbol{\lambda} = 0$ is based on the outcome of the second test. Let η_1 be the asymptotic size of the rank restriction test and η_2 be the asymptotic size of either the normal test or the weighted chi-squared test used in the second stage.

When $\boldsymbol{\alpha}$ is in the span of the column space of $\mathbf{D}_0$ (Table IV), the rank restriction test will accept the null of reduced rank with probability $1-\eta_1$ (asymptotically). Therefore, the probability of using the normal test in the second stage is η_1. Unconditionally, the normal test will reject with probability $p_1 \geq \eta_2$ (in our simulation setup) and the weighted chi-squared test will reject with probability η_2. Therefore, if the two tests are independent, the size of the sequential test is given by

$$\eta_1 p_1 + (1 - \eta_1)\eta_2 \geq \eta_2.$$

In general, the two tests are dependent because both the rank restriction test and the test of $H_0 : \boldsymbol{\alpha}'\boldsymbol{\lambda} = 0$ are specification tests. In this case, we can only establish an upper bound on the probability of rejection of the sequential test, which is given by

$$\eta_1 + \eta_2.$$

When $\boldsymbol{\alpha}$ is not in the span of the column space of $\mathbf{D}_0$ (Table VI), the rank restriction test will reject the null of reduced rank with probability one (asymptotically), so the normal test will be chosen in the second stage. As a result, the asymptotic size of the sequential test is simply η_2.

The results in Tables IV and VI (which are obtained by setting the asymptotic sizes of the first and second tests equal to each other, i.e., $\eta_1 = \eta_2$) show that the proposed sequential test tends to behave well in our simulation setup.

REFERENCES

[1] Ahn, S. C., and C. Gadarowski (2004): "Small Sample Properties of the Model Specification Test Based on the Hansen-Jagannathan Distance," *Journal of Empirical Finance*, 11, 109–132.

Table I
Empirical Size of the Standard Normal Test

Panel A: $\boldsymbol{\alpha} = \mathbf{q} = [1\ ,\mathbf{0}'_{m-1}]'$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.978	0.929	0.689	1.000	1.000	1.000
300	0.977	0.925	0.682	1.000	1.000	1.000
450	0.976	0.923	0.679	1.000	1.000	0.999
600	0.976	0.924	0.679	1.000	1.000	0.999
750	0.975	0.923	0.679	1.000	1.000	0.999
900	0.976	0.923	0.680	1.000	1.000	0.999

Panel B: $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{1}_p$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.968	0.910	0.661	1.000	1.000	0.998
300	0.965	0.907	0.650	1.000	1.000	0.998
450	0.964	0.904	0.650	1.000	1.000	0.998
600	0.965	0.905	0.647	1.000	1.000	0.998
750	0.966	0.904	0.648	1.000	1.000	0.998
900	0.965	0.904	0.648	1.000	1.000	0.997

Panel C: $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{Q}_c^1$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.129	0.071	0.017	0.187	0.115	0.037
300	0.114	0.059	0.013	0.141	0.079	0.020
450	0.109	0.056	0.012	0.127	0.068	0.017
600	0.107	0.055	0.012	0.120	0.063	0.015
750	0.106	0.053	0.011	0.117	0.062	0.014
900	0.105	0.053	0.011	0.115	0.060	0.013

Table II
Empirical Size of the Weighted χ^2 Test

Panel A: $\boldsymbol{\alpha} = \mathbf{q} = [1\ ,\mathbf{0}'_{m-1}]'$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.144	0.082	0.022	0.284	0.189	0.072
300	0.121	0.065	0.015	0.178	0.105	0.031
450	0.115	0.060	0.014	0.151	0.084	0.022
600	0.111	0.057	0.013	0.138	0.074	0.018
750	0.109	0.057	0.012	0.130	0.070	0.016
900	0.107	0.055	0.011	0.125	0.067	0.015

Panel B: $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{1}_p$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.124	0.068	0.018	0.209	0.137	0.052
300	0.111	0.058	0.013	0.136	0.077	0.021
450	0.109	0.057	0.012	0.123	0.066	0.015
600	0.106	0.054	0.012	0.115	0.061	0.014
750	0.105	0.054	0.011	0.112	0.058	0.013
900	0.104	0.054	0.012	0.112	0.058	0.012

Panel C: $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{Q}_c^1$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.132	0.072	0.018	0.185	0.111	0.034
300	0.116	0.061	0.013	0.138	0.076	0.019
450	0.109	0.056	0.012	0.124	0.067	0.016
600	0.108	0.055	0.012	0.119	0.062	0.014
750	0.108	0.054	0.011	0.115	0.060	0.013
900	0.105	0.053	0.010	0.111	0.059	0.013

Table III
Empirical Size of the Rank Test

Panel A: $\boldsymbol{\alpha} = \mathbf{q} = [1\ ,\mathbf{0}'_{m-1}]'$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.095	0.044	0.007	0.069	0.024	0.001
300	0.098	0.048	0.009	0.093	0.044	0.007
450	0.099	0.050	0.009	0.098	0.047	0.008
600	0.099	0.049	0.010	0.099	0.047	0.009
750	0.100	0.050	0.010	0.100	0.049	0.009
900	0.099	0.050	0.010	0.100	0.050	0.009

Panel B: $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{1}_p$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.096	0.045	0.007	0.072	0.026	0.001
300	0.099	0.047	0.009	0.093	0.043	0.007
450	0.100	0.050	0.010	0.098	0.046	0.008
600	0.100	0.050	0.010	0.098	0.048	0.008
750	0.101	0.050	0.010	0.100	0.048	0.009
900	0.101	0.050	0.010	0.100	0.050	0.009

Panel C: $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{Q}_c^1$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.084	0.036	0.004	0.048	0.015	0.001
300	0.093	0.044	0.007	0.079	0.033	0.004
450	0.097	0.046	0.008	0.088	0.039	0.006
600	0.097	0.046	0.008	0.091	0.043	0.007
750	0.097	0.047	0.008	0.094	0.044	0.008
900	0.097	0.048	0.009	0.095	0.045	0.008

Table IV
Empirical Size of the Sequential Test
When $\boldsymbol{\alpha}$ is in the Span of the Column Space of $\mathbf{D}_0$

Panel A: $\boldsymbol{\alpha} = \mathbf{q} = [1\ ,\mathbf{0}'_{m-1}]'$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.145	0.082	0.022	0.284	0.189	0.072
300	0.121	0.065	0.015	0.178	0.105	0.031
450	0.115	0.060	0.014	0.151	0.085	0.022
600	0.111	0.058	0.013	0.138	0.074	0.018
750	0.109	0.057	0.012	0.130	0.070	0.016
900	0.107	0.055	0.011	0.125	0.067	0.015

Panel B: $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{1}_p$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.141	0.072	0.018	0.210	0.137	0.052
300	0.146	0.072	0.014	0.145	0.080	0.021
450	0.149	0.075	0.015	0.143	0.073	0.016
600	0.149	0.074	0.015	0.142	0.072	0.015
750	0.149	0.074	0.015	0.145	0.072	0.015
900	0.149	0.075	0.015	0.147	0.074	0.015

Panel C: $\boldsymbol{\alpha} = \mathbf{D}_0\mathbf{Q}_c^1$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.119	0.067	0.017	0.180	0.110	0.034
300	0.103	0.055	0.012	0.130	0.073	0.019
450	0.095	0.050	0.012	0.116	0.063	0.015
600	0.094	0.049	0.011	0.110	0.058	0.014
750	0.093	0.048	0.010	0.106	0.056	0.013
900	0.091	0.047	0.010	0.102	0.055	0.012

Table V
Empirical Power of the Rank Test

Panel A: $\boldsymbol{\alpha} = \mathbf{1}_m$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.999	0.997	0.965	0.977	0.913	0.531
300	1.000	1.000	1.000	1.000	1.000	1.000
450	1.000	1.000	1.000	1.000	1.000	1.000
600	1.000	1.000	1.000	1.000	1.000	1.000
750	1.000	1.000	1.000	1.000	1.000	1.000
900	1.000	1.000	1.000	1.000	1.000	1.000

Panel B: $\boldsymbol{\alpha} = \sqrt{m}\mathbf{q} + \mathbf{1}_m$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.999	0.997	0.965	0.974	0.904	0.508
300	1.000	1.000	1.000	1.000	1.000	1.000
450	1.000	1.000	1.000	1.000	1.000	1.000
600	1.000	1.000	1.000	1.000	1.000	1.000
750	1.000	1.000	1.000	1.000	1.000	1.000
900	1.000	1.000	1.000	1.000	1.000	1.000

Table VI
Empirical Size of the Sequential Test
When $\boldsymbol{\alpha}$ is not in the Span of the Column Space of $\mathbf{D}_0$

Panel A: $\boldsymbol{\alpha} = \mathbf{1}_m$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.123	0.067	0.022	0.177	0.124	0.130
300	0.110	0.057	0.012	0.132	0.072	0.018
450	0.106	0.054	0.011	0.121	0.065	0.014
600	0.104	0.053	0.011	0.116	0.061	0.013
750	0.104	0.052	0.011	0.112	0.059	0.013
900	0.103	0.051	0.011	0.110	0.057	0.012

Panel B: $\boldsymbol{\alpha} = \sqrt{m}\mathbf{q} + \mathbf{1}_m$

	CAPM			FF3		
	Level of Significance			Level of Significance		
T	10%	5%	1%	10%	5%	1%
150	0.124	0.065	0.022	0.211	0.151	0.142
300	0.110	0.057	0.012	0.151	0.086	0.023
450	0.108	0.054	0.011	0.134	0.073	0.018
600	0.106	0.053	0.011	0.126	0.067	0.016
750	0.105	0.052	0.011	0.119	0.063	0.015
900	0.104	0.052	0.010	0.116	0.061	0.013