Concordia University October 29, 2009

Measure Theory
Midterm

Instructions: No materials, particularly no cell phones and no calculators, are allowed during the examination. There is one page with a total of four problems. Undergraduate students must do Problems 1 – 3. Graduate students must do Problems 1 – 4. Undergraduate students can solve Problem 4 for extra credit, but they should first address the mandatory problems.

(1) (**Lebesgue measure**)
For each of the following statements, find a proof or give a counterexample.
  (a) If E is a bounded, measurable set of real numbers, then $m(E) < \infty$.
  (b) If E is a measurable set of real numbers with $m(E) < \infty$, then E is bounded.
  (c) If $f : E \to (0, \infty)$ is a measurable real-valued function, then $1/f$ is also measurable.
  (d) If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing real-valued function, then f is measurable.

(2) (**Lebesgue integration**)
In each of the following cases, state whether the function f is Lebesgue integrable over the interval I. Justify briefly your answer. (You do not need to calculate the value of the integrals.)
  (a) $I = \mathbb{R}$, $f(x) = x$ if x is rational and $f(x) = 0$ if x is irrational.
  (b) $I = (0, \pi/2)$, $f(x) = \tan x$.
  (c) $I = [1, \infty)$, $f(x) = (-1)^n/n$ if $n \leq x < n + 1$, $n = 1, 2, 3, \ldots$.
  (d) $I = (0, 1)$, $f(x) = n$ if x is of the form $\dfrac{1}{2^n}$ for some integer n and $f(x) = \dfrac{x}{\sin x}$ otherwise.

(3) (a) Let $\{f_n : \mathbb{R} \to [0, \infty)\}_n$ be a non-decreasing sequence of measurable functions, i.e. $f_n \leq f_{n+1}$, $\forall n$. Show that $\displaystyle\int \sup_n f_n = \sup_n \int f_n$.
  (b) Let E be a measurable set with $m(E) < \infty$ and let $\{f_n : E \to [0, \infty)\}_n$ be a sequence of measurable functions bounded uniformly by a positive real number M, i.e. $f_n(x) \leq M$, $\forall x \in E$ and $\forall n$. Show that $\displaystyle\int_E \limsup_n f_n \leq \limsup_n \int_E f_n$.

(4) (**Primarily for graduate students only!**)
  (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f = 0$ a.e., and suppose that f is continuous at a. Show that $f(a) = 0$.
  (b) Is $\chi_{\mathbb{Q}}$ continuous a.e.? Does there exist a continuous function g such that $\chi_{\mathbb{Q}} = g$ a.e.? Explain.
  (c) Is $\chi_{(0,\infty)}$ continuous a.e.? Does there exist a continuous function g such that $\chi_{(0,\infty)} = g$ a.e.? Explain.