MATH 467
Measure Theory
Egoroff's Theorem

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Theorem (A First Version). *If $\{f_n\}_n$ is a sequence of measurable functions converging pointwise to f a.e. on a measurable set E, then $\forall \eta > 0, \exists A \subset E, m(A) < \eta/2$ such that f_n converges uniformly to f on $E \setminus A$.*

Proof. Assume without any loss of generality that f_n converges pointwise to f **everywhere** on E. (Recall similar arguments from the lecture.) Let $\eta > 0$.

For any n and k, positive natural numbers, let

$$E_k^n = \{x \in E : |f_j(x) - f(x)| < \frac{1}{n},\ \forall j > k\}.$$

Fix n and note that $E_k^n \subset E_{k+1}^n$ and $E_k \nearrow E$ as $k \nearrow \infty$. Then, from a lemma of the course, $m(\cup E_k^n) = \lim_{k\to\infty} m(E_k^n) = m(E)$. Therefore there exists a rank k_n such that $m(E \setminus E_k^n) < 1/2^n$ and $|f_j(x) - f(x)| < 1/n$ for all $j > k_n$ and $x \in E_{k_n}^n$.

Now, let N be such that $\displaystyle\sum_{n=N}^{\infty} \frac{1}{2^n} < \eta/2$.

Furthermore, take $A'_\eta = \cap_{n\geq N} E_{k_n}^n$ and note that $m(E \setminus A'_\eta) \leq \displaystyle\sum_{n=N}^{\infty} m(E \setminus E_{k_n}^n) < \eta/2$.

Thus, if given $\delta > 0$, choose $n \geq N$ such that $\frac{1}{n} < \delta$ and note that $x \in A'_\eta$ implies $x \in E_{k_n}^n$, hence $|f_j(x) - f(x)| < \delta$, $\forall j > k_n$. In other words, f_j converges uniformly to f on A'_η. (The set A from the Theorem is the set $E \setminus A'_\eta$.) ■

Theorem (A Second Version). *If $\{f_n\}_n$ is a sequence of measurable functions converging pointwise to f a.e. on a measurable set E, then $\forall \eta > 0$, there exists a closed set $A \subset E, m(A) < \eta$ such that f_n converges uniformly to f on $E \setminus A$.*

Proof. Start with the proof of the first version and choose a closed set $A''_\eta \subset A'_\eta$ with $m(A'_\eta \setminus A''_\eta) < \eta/2$. (We can do this in view of Proposition 3.15, Royden.) Then, $m(E \setminus A''_\eta) < \eta$ and the theorem is established.

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