Solutions to Assignment No. 2, MAST 330

(1) **Problem 14, page 39 (Section 2.1)**: Consider the associated equation $y' = 2y$. Any of its non-trivial (zero) solutions is an integrating factor of the original ODE: $y' + 2y = te^{-2t}$. We solve $\frac{dy}{y} = 2dt$ and obtain $y(t) = e^{2t}$ as a solution. We use it to multiply the given ODE:

$$e^{2t}y' + 2e^{2t}y = te^{-2t}e^{2t} \ \Rightarrow\ e^{2t}y' + 2e^{2t}y = t \ \Rightarrow\ (e^{2t}y)' = t.$$

We integrate with respect to t both sides of the equation:

$$ye^{2t} = \frac{t^2}{2} + C \ \Rightarrow\ y = e^{-2t}\left(\frac{t^2}{2} + C\right),\ C = \text{constant}.$$

Since $y(1) = 0$, we have that C must satisfy the equation: $0 = e^{-2}(1/2 + C)$, thus $C = -1/2$. Thus, the final answer is

$$y = e^{-2t}\left(\frac{t^2}{2} - \frac{1}{2}\right).$$

(2) **Problem 15, page 39 (Section 2.1)**: As $t > 0$, re-write the ODE in the standard form $y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$ and consider the associated equation $y' = (2/t)y$. Any of its non-trivial (zero) solutions is an integrating factor of the original ODE. We solve $\frac{dy}{y} = \frac{2}{t}dt$ and obtain $y(t) = t^2$ as a solution. We use it to multiply the given ODE in standard form:

$$2t^2y' + 2ty = t^3 - t^2 + t \ \Rightarrow\ t^2y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C,\ C = \text{constant}.$$

Thus

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2},\ C = \text{constant}.$$

Since $y(1) = 1/2$, we have that C must satisfy the equation: $\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + \frac{C}{1}$, thus $C = 1/12$. Thus, the final answer is

$$y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}.$$

(3) **Problem 16, page 39 (Section 2.1)**: Consider the associated equation $y' = \frac{2}{t}y$ again whose solution $y = t^2$ is an integrating factor of the linear ODE: $y' + \frac{2}{t}y = \frac{\cos t}{t^2}$. We use t^2 to multiply both sides of the given ODE and obtain:

$$t^2y' + 2ty = \cos t \ \Rightarrow\ (t^2y)' = \cos t \ \Rightarrow\ t^2y = \sin t + C,$$

for $C=$ constant. Thus

$$y=\frac{\sin t}{t^2}+\frac{C}{t^2},\ C=\text{constant}.$$

Since $y(\pi)=0$, we have that C must satisfy the equation: $0=0+C/\pi^2$, thus $C=0$ and the final answer is $y=\dfrac{\sin t}{t^2}$.

(4) **Problem 30, page 40 (Section 2.1)**: Consider the associated equation $y'=-y$ whose solution $y=e^{-t}$ is an integrating factor of the linear ODE: $y'-y=1+3\sin t$. We use e^{-t} to multiply both sides of the given ODE and obtain:

$$e^{-t}y'-e^{-t}y=e^{-t}+3e^{-t}\sin t\ \Rightarrow\ e^{-t}y=-e^{-t}+3\int e^{-t}\sin t\,dt.$$

The last integral can be solved by integration by parts and we obtain: Thus

$$e^{-t}y=-e^{-t}-\frac{3}{2}e^{-t}(\cos t+\sin t)+C,\ C=\text{constant}$$

or

$$y=-1-\frac{3}{2}(\cos t+\sin t)+Ce^t,\quad C=\text{constant}. \tag{1}$$

Note that, if $C\neq 0$, the term Ce^t becomes unbounded as $t\to\infty$, thus we must y_0 such that the initial condition $y(0)=y_0$ implies $C=0$. From (1), we have that $y(0)=-1-3/2+C$, thus, if $C=0$, $y_0=-5/2$.

(5) **Problem 3, page 47 (Section 2.2)**: We start with $y'+y^2\sin x=0\ \Rightarrow\ y'=-y^2\sin x$. We would like to divide by y, but first we must check if $y=0$, for all x, is a solution. Indeed, it is. So one solution of the given ODE is the constant function zero. Next, we have $\dfrac{dy}{y^2}=-\sin x\,dx\quad\Rightarrow\quad -\dfrac{1}{y}=\cos x+C$, where C is arbitrary constant. Explicitly, $y=-\dfrac{1}{\cos x+C}$, C arbitrary constant, and $y=0$ are all solutions to the ODE $y'+y^2\sin x=0$.

(6) **Problem 4, page 47 (Section 2.2)**: We have $(3+2y)\,dy=(3x^2-1)\,dx$ which, by integration, gives the implicit solution $3y+y^2=x^3-x+C$, where C is arbitrary constant. (For each fixed C, this is the equation of a curve which should not pass through any point of y-coordinate $y=-3/2$, because the ODE is undefined for that value of y.)

(7) **Problem 5, page 47 (Section 2.2)**: Before dividing the equation by $\cos^2 2y$, we should note that each constant function $y=\pi/4+n\pi/2$, n arbitrary integer, is a solution to the ODE because it makes both sides of the equation zero.

Assuming now that y is different from any of the values above, we re-write the ODE in the form

$$\frac{dy}{\cos^2 2y} = \cos^2 x\,dx \;\Rightarrow\; \frac{1}{2}\sec^2(2y)\,dy = \cos^2 x$$

$$\Rightarrow\; \frac{1}{2}\tan(2y) = \int \cos^2 x\,dx + C.$$

We will now evaluate

$$\int \cos^2 x\,dx = \int \frac{\cos(2x)-1}{2}\,dx = \frac{1}{4}\sin(2x) - \frac{x}{2} + \text{constant}.$$

Thus, we obtain the non-constant implicit solutions:

$$\frac{1}{2}\tan(2y) = \frac{1}{4}\sin(2x) - \frac{x}{2} + C,\; C = \text{arbitrary constant}.$$

(8) **Problem 6, page 47 (Section 2.2)**: We note that $y = \pm 1$ are each solutions to the ODE for all real x's.

If $y \neq \pm 1$, then we can re-write the ODE as

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{x} \;\Rightarrow\; \arcsin y = \ln|x| + C,\; C = \text{arbitrary constant}.$$

(These implicit solutions are not defined at $x = 0$, which is obvious from the fact that ln is not defined at zero.)

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